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A sharp Hölder estimate for elliptic equations in two variables

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Abstract

We prove a sharp Hölder estimate for solutions of linear two-dimensional, divergence form elliptic equations with measurable coefficients, such that the matrix of the coefficients is symmetric and has *unit determinant*. Our result extends some previous work by Piccinini and Spagnolo [7]. The proof relies on a sharp Wirtinger type inequality.

Key Words: linear elliptic equation, measurable coefficients, Hölder regularity

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1 Introduction

Let Ω be a bounded open subset of $\mathbb{R}^n$ and let $u \in H^1_{\rm loc}(\Omega)$ be a weak solution to the linear elliptic equation in divergence form

$$(a_{ij}u_{x_i})_{x_j} = 0 \qquad \text{in } \Omega, \tag{1}$$

where a_{ij}, $i, j = 1, 2, \ldots, n$, are bounded measurable functions in Ω satisfying the ellipticity condition

$$\lambda|\xi|^2 \le a_{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2, \tag{2}$$

for all $x \in \Omega$, for all $\xi \in \mathbb{R}^n$ and for some $0 < \lambda \le \Lambda$. It is well known (see, e.g., [1, 2, 3, 6] and references therein) that solutions to (1) are α-Hölder continuous in Ω for some $0 < \alpha < 1$. More precisely, for every compact subset $K \subset\subset \Omega$ there holds

$$\sup_{x,y\in K, x\neq y} \frac{|u(x) - u(y)|}{|x - y|^\alpha} < +\infty. \tag{3}$$

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Moreover, estimates for the Hölder exponent α may be obtained, depending on the ellipticity constant $L = \Lambda/\lambda$ only. In this note we consider the case $n = 2$, which unless otherwise stated we assume henceforth. For this case, sharp Hölder estimates were obtained by Piccinini and Spagnolo in [7] under the symmetry assumption

$$a_{ij}(x) = a_{ji}(x) \qquad \text{in } \Omega. \tag{4}$$

We collect in the following theorem some results from [7] which are relevant to our considerations.

Theorem 1 (Piccinini and Spagnolo [7]). *Let $u \in H^1_{\mathrm{loc}}(\Omega)$ be a weak solution to* (1)*.*

(i) If the coefficients a_{ij} satisfy (2) *and* (4)*, then $\alpha = L^{-1/2}$.*

(ii) If the coefficients a_{ij} satisfy (2)*,* (4) *and if furthermore $a_{ij} = a(x)\delta_{ij}$ for some measurable function $a = a(x)$ satisfying $\lambda \le a(x) \le \Lambda$ for all $x \in \Omega$, then $\alpha = \frac{4}{\pi}\arctan L^{-1/2}$.*

These values of α are sharp.

In particular, Theorem 1–(ii) implies that the optimal value of α increases by restricting the matrices A to the class of isotropic matrices $A = a(x)I$. Thus, it is natural to seek other classes of matrices A for which the corresponding optimal value of α may be improved. Our main result in this note is to show that if $A = (a_{ij})$ satisfies (2), (4) and if furthermore A has *unit determinant*, namely, if

$$\det A(x) = a_{11}(x)a_{22}(x) - a_{12}(x)a_{21}(x) \equiv 1 \qquad \text{in } \Omega, \tag{5}$$

then a more accurate *integral* characterization of α may be obtained. Moreover, our result is *sharp* within the class of matrices satisfying (2), (4) and (5). It should be mentioned that condition (5) is relevant in the context of quasiharmonic fields, see [5].

Theorem 2 (Main result). *Let $A = (a_{ij})$ satisfy* (2)*,* (4) *and* (5) *in Ω and let u satisfy* (1)*. Then u is α-Hölder continuous in Ω, with α given by*

$$\alpha = 2\pi \left(\sup_{x_0\in\Omega} \operatorname{ess\,sup}_{0<r<\operatorname{dist}(x_0,\partial\Omega)} \int_{|\xi|=1} a_{ij}(x_0 + r\xi)\xi_i\xi_j \, \mathrm{d}\sigma_\xi \right)^{-1}. \tag{6}$$

We note that under assumption (5), we may choose $\lambda = 1/\Lambda$ in (2) and therefore the ellipticity constant takes the value $L = \Lambda^2$. Hence, the Piccinini-Spagnolo estimate in Theorem 1–(i) yields in this case $\alpha = \Lambda^{-1}$. On the other hand, recalling that $\Lambda = \sup_{x\in\Omega} \sup_{|\xi|=1} a_{ij}(x)\xi_i\xi_j$, it is clear that $\alpha \ge \Lambda^{-1}$.

Theorem 2 implies the following

Corollary 1. *Let $A = (a_{ij})$ satisfy* (2)*,* (4) *and* (5) *and let u satisfy* (1)*. Then the least upper bound for the admissible values of the Hölder exponent of u is given by*

$$\bar{\alpha} = 2\pi \left(\sup_{x_0\in\Omega} \inf_{0<r_0<\operatorname{dist}(x_0,\partial\Omega)} \operatorname{ess\,sup}_{0<r<r_0} \int_{|\xi|=1} a_{ij}(x_0 + r\xi)\xi_i\xi_j \, \mathrm{d}\sigma \right)^{-1}. \tag{7}$$

Theorem 2 is *sharp*, in the sense of the following

Example 1 (Sharpness). *Let $\Omega = B$ the unit ball in $\mathbb{R}^2$, let $\theta = \arg x$ and let*

$$A(x) = \frac{1}{k(\theta)} I + \left(k(\theta) - \frac{1}{k(\theta)} \right) \frac{x \otimes x}{|x|^2} \qquad \text{in } B \setminus \{0\}. \tag{8}$$

where $k : \mathbb{R} \to \mathbb{R}^+$ is a 2π-periodic measurable function bounded from above and away from 0. Then $\det A(x) \equiv 1$*. By a suitable choice of k, we may obtain that*

$$\bar{\alpha} = 2\pi \left(\int_0^{2\pi} k \right)^{-1}. \tag{9}$$

On the other hand the function $u \in H^1(B)$ defined by

$$u(x) = |x|^{\bar{\alpha}} \cos \left(\bar{\alpha} \int_0^{\arg x} k \right) \tag{10}$$

satisfies equation (1) *with A given by* (8)*. Clearly, its Hölder exponent is exactly $\bar{\alpha}$.*

A verification of Example 1 is provided in the Appendix. In a forthcoming note, we will show that the functions defined by (10) are also of interest in the context of quasiconformal maps. The remaining part of this note is devoted to the proof of Theorem 2.

Notation

For a fixed $x_0 \in \Omega$, let $x = x_0 + \rho e^{i\theta}$, be the polar coordinate transformation centered at x_0. We denote $u_{x_i} = \partial u / \partial x_i$, $u_\rho = \partial u / \partial \rho$, $u_\theta = \partial u / \partial \theta$. Following the notation in [7], we denote $\nabla u = (u_{x_1}, u_{x_2})$ and $\overline{\nabla} u = (u_\rho, \frac{u_\theta}{\rho})$. We denote by $J(\theta)$ the rotation matrix

$$J(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

In this notation, we have

$$\nabla u = J(\theta) \overline{\nabla} u. \tag{11}$$

Finally, we denote by $\langle \cdot, \cdot \rangle$ the Euclidean scalar product in $\mathbb{R}^2$.

2 Proof of Theorem 2

In the spirit of [7], a key ingredient in the proof of Theorem 2 is a sharp Wirtinger type inequality.

Lemma 1 (Sharp Wirtinger type inequality). *Let $a : \mathbb{R} \to \mathbb{R}$ be a 2π-periodic measurable function which is bounded from above and away from 0. Then,*

$$\int_0^{2\pi} a w^2 \leq \left(\frac{1}{2\pi} \int_0^{2\pi} a \right)^2 \int_0^{2\pi} \frac{1}{a} (w')^2 \tag{12}$$

for every $w \in H^1_{\mathrm{loc}}(\mathbb{R})$ such that w is 2π-periodic and satisfies $\int_0^{2\pi} aw = 0$. Equality in (12) is attained if and only if w is of the form

$$w(\theta) = C\cos\left(\frac{2\pi}{\int_0^{2\pi} a}\int_0^\theta a + \varphi\right) \tag{13}$$

for some $C \neq 0$ and $\varphi \in \mathbb{R}$.

Proof. Let

$$X = \left\{w \in H^1_{\mathrm{loc}}(\mathbb{R}) :\ w \text{ is } 2\pi\text{−periodic}\right\}$$

and consider the functional I defined by

$$I(w) = \frac{\int_0^{2\pi} \frac{1}{a}(w')^2}{\int_0^{2\pi} aw^2}$$

for all $w \in X \setminus \{0\}$. We equivalently have to show that

$$\inf\left\{I(w) :\ w \in X \setminus \{0\}, \int_0^{2\pi} aw = 0\right\} = \left(\frac{2\pi}{\int_0^{2\pi} a}\right)^2$$

and that the infimum is attained exactly on functions of the form (13). By Sobolev embeddings and standard compactness arguments, there exists $\underline{w} \in X \setminus \{0\}$ satisfying $\int_0^{2\pi} a\underline{w} = 0$ and such that

$$I(\underline{w}) = \inf\left\{I(w) :\ w \in X, \int_0^{2\pi} aw = 0\right\} > 0.$$

By the minimum property of $\underline{w}$ we have that $I'(\underline{w})\left(\phi - (2\pi)^{-1}\int_0^{2\pi} a\phi\right) = 0$ for all $\phi \in X$. By properties of $\underline{w}$ it follows that

$$\int_0^{2\pi} \frac{1}{a}\underline{w}'\phi' - I(\underline{w})\int_0^{2\pi} a\underline{w}\phi = 0 \qquad \forall \phi \in X.$$

Let $\Theta = \int_0^\theta a$, $w(\theta) = W(\Theta)$, $\phi(\theta) = \Phi(\Theta)$. Then $W, \Phi \in H^1_{\mathrm{loc}}(\mathbb{R})$ are $\left(\int_0^{2\pi} a\right)$-periodic and

$$\int_0^{\int_0^{2\pi} a} W'(\Theta)\Phi'(\Theta)\,\mathrm{d}\Theta = I(\underline{w})\int_0^{\int_0^{2\pi} a} W(\Theta)\Phi(\Theta)\,\mathrm{d}\Theta.$$

Equivalently, W is a weak solution for

$$-(W')' = I(\underline{w})W.$$

It follows that $I(\underline{w}) = (2\pi/\int_0^{2\pi} a)^2$ and

$$W(\Theta) = C\cos(\frac{2\pi}{\int_0^{2\pi} a}\Theta + \varphi)$$

for some $C \neq 0$ and $\varphi \in \mathbb{R}$. Recalling the definition of W and Θ, we conclude the proof. $\square$

We can now provide the

Proof of Theorem 2. For every $0 < r < \mathrm{dist}(x_0, \partial\Omega)$ let

$$g_{x_0}(r) = \int_{|x-x_0|<r} a_{ij} u_{x_i} u_{x_j} \,\mathrm{d}x.$$

It is well-known (see, e.g., [2, 3, 4]) that a sufficient condition for (3) is that the function $G_{x_0}(r) = r^{-2\alpha} g_{x_0}(r)$ be bounded in r, uniformly on compact subsets with respect to x_0. We consider the matrix $P = (p_{ij})$ defined in polar coordinates $x = x_0 + \rho e^{i\theta}$ by

$$P(x_0 + \rho e^{i\theta}) = J(\theta)^* A(x_0 + \rho e^{i\theta}) J(\theta), \tag{14}$$

where J is the rotation matrix defined in the Introduction. Then, by (11) we have

$$\begin{aligned} a_{ij} u_{x_i} u_{x_j} =& \langle A\nabla u, \nabla u\rangle = \langle AJ\overline{\nabla} u, J\overline{\nabla} u\rangle = \langle P\overline{\nabla} u, \overline{\nabla} u\rangle \\ =& p_{11} u_\rho^2 + (p_{12} + p_{21})\, u_\rho \frac{u_\theta}{\rho} + p_{22} \left(\frac{u_\theta}{\rho}\right)^2. \end{aligned}$$

Note that $\det P = 1$ and $P = P^*$. Therefore, denoting $p_{11} = p$, $p_{12} = p_{21} = q$, $p_{22} = (1+q^2)/p$, we can write

$$a_{ij} u_{x_i} u_{x_j} = p u_\rho^2 + 2q u_\rho \frac{u_\theta}{\rho} + \frac{1+q^2}{p} \left(\frac{u_\theta}{\rho}\right)^2.$$

In view of (1) for every $k \in \mathbb{R}$ we have $a_{ij} u_{x_i} u_{x_j} = ((u-k) a_{ij} u_{x_i})_{x_j}$ and therefore by the divergence theorem and an approximation argument (see [7])

$$g_{x_0}(r) = \int_{S_r} (u-k) a_{ij} u_{x_i} n_j \,\mathrm{d}\sigma$$

where S_r is the circle of center x_0 and radius r, and n is the outward normal to S_r. Since $n = (\cos\theta, \sin\theta) = Je_1$ we have

$$a_{ij} u_{x_i} n_j = \langle A\nabla u, n\rangle = \langle AJ\overline{\nabla} u, Je_1\rangle = \langle P\overline{\nabla} u, e_1\rangle = p u_\rho + q\frac{u_\theta}{\rho}$$

and therefore by Hölder's inequality

$$\begin{aligned} g_{x_0}(r) =& \int_{S_r} (u-k) \left(p u_\rho + q\frac{u_\theta}{\rho}\right) \mathrm{d}\sigma = \int_{S_r} \sqrt{p}(u-k) \left(\sqrt{p} u_\rho + \frac{q}{\sqrt{p}} \frac{u_\theta}{\rho}\right) \mathrm{d}\sigma \\ \leq& \left[\int_{S_r} p(u-k)^2 \,\mathrm{d}\sigma \int_{S_r} \left(\sqrt{p} u_\rho + \frac{q}{\sqrt{p}} \frac{u_\theta}{\rho}\right)^2 \mathrm{d}\sigma\right]^{1/2}. \end{aligned}$$

Choosing $k = |S_r|^{-1} \int_{S_r} pu \,\mathrm{d}\sigma$, rescaling and using (12) with $a(\theta) = p(x_0 + re^{i\theta})$, $w(\theta) = u(x_0 + re^{i\theta}) - k$, we obtain the inequality

$$\int_{S_r} p(u-k)^2 \,\mathrm{d}\sigma \leq \left(|S_r|^{-1} \int_{S_r} p \,\mathrm{d}\sigma\right)^2 \int_{S_r} \frac{1}{p} u_\theta^2 \,\mathrm{d}\sigma.$$

It follows that

$$g_{x_0}(r) \le \left(|S_r|^{-1}\int_{S_r} p\,\mathrm{d}\sigma\right)\left[\int_{S_r}\frac{1}{p}u_\theta^2\,\mathrm{d}\sigma\right]^{1/2}\left[\int_{S_r}\left(\sqrt{p}u_\rho+\frac{q}{\sqrt{p}}\frac{u_\theta}{\rho}\right)^2\mathrm{d}\sigma\right]^{1/2}.$$

Equivalently, we may write

$$\begin{aligned}g_{x_0}(r) \le &\left(|S_r|^{-1}\int_{S_r} p\,\mathrm{d}\sigma\right) r\\ &\times\left[\int_{S_r}\frac{1}{p}\left(\frac{u_\theta}{\rho}\right)^2\mathrm{d}\sigma\right]^{1/2}\left[\int_{S_r}\left(\sqrt{p}u_\rho+\frac{q}{\sqrt{p}}\frac{u_\theta}{\rho}\right)^2\mathrm{d}\sigma\right]^{1/2}.\end{aligned}$$

By the inequality $\sqrt{ab}\le\frac{1}{2}(a+b)$ for every $a,b>0$ we obtain for a.e. $0<r<\operatorname{dist}(x_0,\partial\Omega)$ that

(15)

$$\begin{aligned}g_{x_0}(r) \le&\frac{1}{2}\left(|S_r|^{-1}\int_{S_r} p\,\mathrm{d}\sigma\right) r\int_{S_r}\left[\frac{1}{p}\left(\frac{u_\theta}{\rho}\right)^2+\left(\sqrt{p}u_\rho+\frac{q}{\sqrt{p}}\frac{u_\theta}{\rho}\right)^2\right]\mathrm{d}\sigma\\ =&\frac{1}{2}\left(|S_r|^{-1}\int_{S_r} p\,\mathrm{d}\sigma\right) r\int_{S_r}\left[pu_\rho^2+2qu_\rho\frac{u_\theta}{\rho}+\left(\frac{1+q^2}{p}\right)\left(\frac{u_\theta}{\rho}\right)^2\right]\mathrm{d}\sigma\\ =&\frac{1}{2}\left(|S_r|^{-1}\int_{S_r} p\,\mathrm{d}\sigma\right) r\int_{S_r}\langle P\overline{\nabla}u,\overline{\nabla}u\rangle\\ =&\frac{1}{2}\left(|S_r|^{-1}\int_{S_r} p\,\mathrm{d}\sigma\right) r\int_{S_r}\langle A\nabla u,\nabla u\rangle\\ \le&\frac{1}{2}\operatorname{ess\,sup}_{0<r<\operatorname{dist}(x_0,\partial\Omega)}\left(|S_r|^{-1}\int_{S_r} p\,\mathrm{d}\sigma\right) r\int_{S_r}\langle A\nabla u,\nabla u\rangle\\ \le&\frac{1}{2}\sup_{x_0\in\Omega}\operatorname{ess\,sup}_{0<r<\operatorname{dist}(x_0,\partial\Omega)}\left(|S_r|^{-1}\int_{S_r} p\,\mathrm{d}\sigma\right) r\int_{S_r}\langle A\nabla u,\nabla u\rangle.\end{aligned}$$

Finally, we note that for all $\xi\in\mathbb{R}^2$, $|\xi|=1$, we have $\xi=J(\theta)e_1$, $\theta=\arg\xi$, and therefore

$$\begin{aligned}p(x_0+r\xi)=&\langle P(x_0+r\xi)e_1,e_1\rangle=\langle J^*A(x_0+r\xi)Je_1,e_1\rangle\\ =&\langle A(x_0+r\xi)\xi,\xi\rangle=a_{ij}(x_0+r\xi)\xi_i\xi_j.\end{aligned}\tag{16}$$

Consequently, we have

$$\begin{aligned}&\sup_{x_0\in\Omega}\operatorname{ess\,sup}_{0<r<\operatorname{dist}(x_0,\partial\Omega)}\left(|S_r|^{-1}\int_{S_r} p\,\mathrm{d}\sigma\right)\\ &=(2\pi)^{-1}\sup_{x_0\in\Omega}\operatorname{ess\,sup}_{0<r<\operatorname{dist}(x_0,\partial\Omega)}\int_{|\xi|=1}a_{ij}(x_0+r\xi)\xi_i\xi_j\,\mathrm{d}\sigma\\ &=\alpha^{-1}.\end{aligned}$$

We note that g_{x_0} is differentiable almost everywhere and that

$$g'_{x_0}(r)=\int_{S_r} p\,\mathrm{d}\sigma \qquad \text{a.e. } r.$$

Therefore, (15) yields

$$g_{x_0}(r) \le \frac{r g'_{x_0}(r)}{2\alpha}. \tag{17}$$

Recalling that $G_{x_0}(r) = r^{-2\alpha} g_{x_0}(r)$, we obtain from (17) that $(\log G_{x_0}(r))' \ge 0$ a.e. It follows that

$$G_{x_0}(r) \le G_{x_0}(\mathrm{dist}(x_0, \partial\Omega)) - \int_r^{\mathrm{dist}(x_0,\partial\Omega)} G'_{x_0}(\rho)\,\mathrm{d}\rho \le G_{x_0}(\mathrm{dist}(x_0, \partial\Omega)).$$

Hence, the desired Hölder estimate is established. $\square$

Appendix

We have postponed to this appendix a

Verification of Example 1. In polar coordinates $x = \rho e^{i\theta}$, we define $K(\theta) = \mathrm{diag}(k(\theta), 1/k(\theta))$. Then the matrix A defined in (8) may be equivalently written in the form

$$\begin{aligned} A(x) =& J(\theta)K(\theta)J^*(\theta) \\ =& \begin{pmatrix} k(\theta)\cos^2\theta + \frac{1}{k(\theta)}\sin^2\theta & \left(k(\theta) - \frac{1}{k(\theta)}\right)\sin\theta\cos\theta \\ \left(k(\theta) - \frac{1}{k(\theta)}\right)\sin\theta\cos\theta & k(\theta)\sin^2\theta + \frac{1}{k(\theta)}\cos^2\theta \end{pmatrix}, \end{aligned} \tag{18}$$

where

$$J(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Hence, it is clear that $\det A \equiv 1$. In order to check (9), we note that if $x_0 = 0$ we have, for all $0 < r < 1$:

$$a_{ij}(r\xi)\xi_i\xi_j = \langle A(r\xi)\xi, \xi\rangle = \langle J^*(\theta)A(r\xi)J(\theta)e_1, e_1\rangle = \langle K(\theta)e_1, e_1\rangle = k(\theta),$$

where $\theta = \arg\xi$. Therefore,

$$(2\pi)^{-1}\int_{|\xi|=1} a_{ij}(r\xi)\xi_i\xi_j\,\mathrm{d}\sigma_\xi = (2\pi)^{-1}\int_0^{2\pi} k, \tag{19}$$

for all $0 < r < 1$. We assume that k is smooth. Then, for $x_0 \ne 0$, we have that A is smooth near x_0. It follows that

$$\inf_{0<r_0<\mathrm{dist}(x_0,\partial B)} \operatorname{ess\,sup}_{0<r<r_0} \int_{|\xi|=1} a_{ij}(x_0 + r\xi)\xi_i\xi_j\,\mathrm{d}\sigma_\xi = \int_{|\xi|=1} a_{ij}(x_0)\xi_i\xi_j\,\mathrm{d}\sigma_\xi.$$

In view of (18), we compute

$$\begin{aligned} \int_{|\xi|=1} a_{ij}(x_0)\xi_i\xi_j\,\mathrm{d}\sigma_\xi =& \int_{|\xi|=1} \langle A(x_0)\xi, \xi\rangle\,\mathrm{d}\sigma_\xi \\ =& \int_{|\xi|=1} \langle K(\theta_0)J^*(\theta_0)\xi, J^*(\theta_0)\xi\rangle\,\mathrm{d}\sigma_\xi \\ =& \int_0^{2\pi} \left\{k(\theta_0)\cos^2(\varphi) + \frac{1}{k(\theta_0)}\sin^2(\varphi)\right\}\,\mathrm{d}\varphi \\ =& \pi\left\{k(\theta_0) + \frac{1}{k(\theta_0)}\right\}, \end{aligned}$$

where $\theta_0 = \arg x_0$. Therefore, for every $x_0 \neq 0$ we obtain that

$$(2\pi)^{-1} \inf_{0<r_0<\mathrm{dist}(x_0,\partial B)} \operatorname{ess\,sup}_{0<r<r_0} \int_{|\xi|=1} a_{ij}(x_0 + r\xi)\xi_i\xi_j \,\mathrm{d}\sigma_\xi = \frac{1}{2}\left\{k(\arg x_0) + \frac{1}{k(\arg x_0)}\right\}. \tag{20}$$

In view of (19) and (20), formula (7) takes the form:

$$\bar{\alpha}^{-1} = \max\left\{(2\pi)^{-1}\int_0^{2\pi} k, \sup_{x_0\neq 0} \frac{1}{2}\left(k(\arg x_0) + \frac{1}{k(\arg x_0)}\right)\right\}.$$

Choosing $k(\theta)$ such that $M \leq k \leq 3M/2$, for $M \gg 1$ we achieve

$$\bar{\alpha} = 2\pi\left(\int_0^{2\pi} k\right)^{-1}.$$

On the other hand, it is readily checked that u given by (10) is a weak solution to the equation

$$\left(\rho k(\theta)u_\rho\right)_\rho + \left(\frac{1}{k(\theta)}\frac{u_\theta}{\rho}\right)_\theta = 0, \tag{21}$$

i.e.,

$$\int_B \left(k(\theta)u_\rho v_\rho + \frac{1}{k(\theta)}\frac{u_\theta}{\rho}\frac{v_\theta}{\rho}\right)\rho \,\mathrm{d}\rho\,\mathrm{d}\theta = 0$$

for all $v \in H^1(B)$ compactly supported in B. Recalling (11) and the definition of K, we have

$$\begin{aligned} 0 &= \int_B \langle K\overline{\nabla}u, \overline{\nabla}u\rangle\rho\,\mathrm{d}\rho\,\mathrm{d}\theta = \int_B \langle KJ^*\nabla u, J^*\nabla u\rangle\,\mathrm{d}x \\ &= \int_B \langle JKJ^*\nabla u, \nabla u\rangle\,\mathrm{d}x = \int_B \langle A\nabla u, \nabla u\rangle\,\mathrm{d}x. \end{aligned}$$

Therefore, in cartesian coordinates, equation (21) takes the form (1) with $A = (a_{ij})$ given by (18). □

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